Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On the Conformal Geometry of Analytic Arcs.[*]

By G. A. PFEIFFER.

Introduction.

Conformal geometry is the geometry associated with the group of conformal transformations. When only the plane of real points is considered, a non-singular conformal transformation is given by any analytic function of a single complex variable,

$$Z=f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots,$$

where $a_1 \neq 0$. When the plane of complex points is the fundamental domain, the general non-singular conformal transformation is of the form

$$U=f(u)=a_0+a_1(u-u_0)+a_2(u-u_0)^2+\ldots,$$
$$V=g(v)=b_0+b_1(v-v_0)+b_2(v-v_0)^2+\ldots,$$

where $u=x+iy$ and $v=x-iy$, x and y, which may be real or complex, being the Cartesian coordinates of a point; also, $f(u)$ and $g(v)$ have radii of convergence different from zero and $a_1 \neq 0$, $b_1 \neq 0$. Both the real and complex, or general, conformal groups are infinite continuous groups in the Lie terminology. Real and complex conformal geometries consist of the codifications of the invariants and covariants of configurations under these respective groups.

In the consideration of harmonic continuation in function theory the notion of general or Schwarzian symmetry plays an important rôle. This notion was introduced by Schwarz in his work, "Ueber die Integration der partiellen Differentialgleichung $\Delta u=0$ unter vorgeschriebenen Grenz- und Unstetigkeitsbedingungen," which was written in 1870.[†] This notion is, naturally, also important in conformal geometry for the same reason that it is in the above discussion, viz.: Schwarzian symmetry is a covariant property of an analytic curve under the conformal group. The usefulness of this property is well illustrated in the method employed by Kasner[‡] in finding conformal invariants of two intersecting analytic arcs which form an angle whose magnitude is commensurable with π.

---

[*] Read before the American Mathematical Society, October 31, 1914.

[†] H. A. Schwarz, "Werke," Vol. II, p. 151.

[‡] E. Kasner, "Conformal Geometry," *Proceedings of the International Congress of Mathematicians*, p. 82, Cambridge, 1912. See p. 422 below.

A good part of the following is devoted to an investigation of the properties of the transformation of general symmetry, or, briefly, symmetry, *i. e.*, the pairing of points symmetric with respect to an analytic arc, and the associated functional equations. In particular, the existence of an analytic arc, called a *symmetric bisector*, such that one of two intersecting analytic arcs is the symmetric image of the other with respect to the bisector is considered. This is called the *bisection problem.** Taking the two intersecting analytic arcs to be given by

$$y=y_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots \text{ and } y=y_0+b_1(x-x_0)+b_2(x-x_0)^2+\ldots,$$

it is found that if $\left|\frac{1-a_1i}{1+a_1i}\cdot\frac{1+b_1i}{1-b_1i}\right| \neq 0$ or 1, there exist just two symmetric bisectors. If $\left|\frac{1-a_1i}{1+a_1i}\cdot\frac{1+b_1i}{1-b_1i}\right|=1$, it is found necessary to consider two cases: firstly, where $\left(\frac{1-a_1i}{1+a_1i}\cdot\frac{1+b_1i}{1-b_1i}\right)^n \neq 1$, $n=$ any positive integer; and secondly, where $\left(\frac{1-a_1i}{1+a_1i}\cdot\frac{1+b_1i}{1-b_1i}\right)^n=1$, n being some positive integer. In the former case just two formal solutions for a bisector exist, while in the latter it is shown that, in case the given configuration is not conformally equivalent to a pair of intersecting rectilinear segments, no more than two formal solutions exist and the conditions which determine the number of formal solutions are given; if, in this latter case, the given configuration is conformally equivalent to a pair of intersecting rectilinear segments, it is shown that the configuration has an infinity of symmetric bisectors. The case when any of the numbers $1\pm a_1i$, $1\pm b_1i$ is zero is not considered; in this case either a_1 or b_1 is $\pm i$; *i. e.*, the slope of at least one of the arcs at the common point is minimal, and the symmetric transformation concerned is singular. The uniqueness of a symmetric n-sector of two intersecting analytic arcs with a definite coefficient of the first degree term, *i. e.*, with a definite slope at the point of intersection of the given arcs, when n is a power of 2, is also demonstrated.

Out of the above investigation certain other important theorems in conformal geometry are obtained, two relating to the conformal equivalence of a pair of intersecting analytic arcs to a pair of intersecting rectilinear segments, and another concerning the existence of a unique absolute conformal invariant

---

* So called by Kasner, *loc. cit.*

of a pair of intersecting analytic arcs which are such that some positive integral power of $\frac{1-a_1 i}{1+a_1 i}\cdot\frac{1+b_1 i}{1-b_1 i}$ is unity, or, in case the real plane is fundamental, which are such that the magnitude of the angle formed by them is commensurable with π. Two cases of this theorem, first, in which the two arcs are tangent, and second, in which the configuration, in normal form, consists of the analytic arc C: $x=b_2y^2+b_3y^3+\ldots$, $b_3\neq 0$, and the segment of the axis of x in the vicinity of the origin, have been proved by Kasner.*

Properties of transformations which are the products of two and three symmetric transformations are investigated; and theorems concerning necessary and sufficient conditions that the product of two symmetric transformations be a non-singular direct conformal transformation, and that the product of three symmetric transformations be another symmetric transformation, are given.

§ 1. *Symmetry in the Plane of Real Points.*

Definition. Let C be a real curve, analytic in the neighborhood of the point (x_0, y_0), and let T be a real non-singular conformal transformation which transforms the arc of the curve C in the neighborhood of the point (x_0, y_0) into the rectilinear segment Γ. If P and P' are two points which are symmetric with respect to the straight line segment Γ, then the points Q and Q' which correspond to the points P and P' under the transformation T are said to be *symmetric with respect to the analytic arc.*†

The above definition is the function-theoretic definition of Schwarz, and the real plane of analysis is understood to be the fundamental domain. In the discussion immediately following, the real plane is that under consideration. The above definition is of value because it is shown‡ that if Q and Q' are symmetric with respect to an analytic arc C for the non-singular conformal transformation T, they are also symmetric with respect to the analytic arc C for any non-singular conformal transformation which transforms the analytic arc C into a rectilinear segment. Thus, given an analytic arc, we have in a certain vicinity of a point on the arc an involutoric correspondence in which each point in that vicinity corresponds to the point which is symmetric to it with respect to the arc. This correspondence is called the symmetric transformation or, more

---

* Kasner, *loc. cit.*, pp. 84, 86.

† A real curve is analytic in the neighborhood of the point (x_0, y_0) if the equation of the locus of its points in that neighborhood is of the form $y=y_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots$ or $x=x_0+a_1(y-y_0)+a_2(y-y_0)+\ldots$, the power series having a radius of convergence different from zero. The portion of the curve given by either development is called an *analytic arc.*

‡ See Osgood, "Lehrbuch der Funktionentheorie," 2d ed., p. 672.

briefly, the symmetry determined by the analytic arc. It is then seen that if T is a non-singular conformal tranformation which transforms an analytic arc into a segment of a straight line, the symmetry determined by the arc is TST^{-1}, where S is the reflection with respect to the straight line which may and will be taken as the x-axis, in which case S is given by the equation

$$Z=\bar{z},$$

where $\bar{z}$ denotes the conjugate of z. Consequently, a symmetry is a non-singular reversed conformal transformation and, further, each point of the determining analytic arc is transformed into itself by the transformation of symmetry. Conversely, any non-singular reversed conformal transformation which transforms each point of an analytic arc into itself is a symmetry determined by the arc, for any non-singular reversed conformal transformation T' may be written $T'=TST^{-1}$, where T is a given non-singular conformal transformation and S is a non-singular reversed conformal transformation. Now, if T be taken as a non-singular conformal transformation which transforms the analytic arc C into the axis of abscissas, then, in order that each point of C be transformed into itself, it is necessary that S be the reflection with respect to the x-axis; *i. e.*, T' must be the symmetric transformation determined by the analytic arc C.

The equation of the symmetry determined by the analytic arc C: $y=a_1x+a_2x^2+\ldots.$* is readily gotten from the fact just stated that the symmetry determined by a given analytic arc is the non-singular reversed conformal transformation which leaves each point of the given analytic arc invariant. Thus, let the symmetry determined by the analytic arc C be given by

$$Z=A_1\bar{z}+A_2\bar{z}^2+A_3\bar{z}^3+\ldots., \qquad A_1\neq 0. \tag{1}$$

Since the point (x, y) of the arc C corresponds to itself under the transformation (1), we have the identity

$$x+iy=A_1(x-iy)+A_2(x-iy)^2+\ldots.,$$

or

$$\begin{aligned}(1+a_1i)x+a_2ix^2+a_3ix^3+\ldots.&\\ =A_1[(1-a_1i)x-a_2ix^2-a_3ix^3-\ldots.]&\\ +A_2[(1-a_1i)x-a_2ix^2-a_3ix^3-\ldots.]^2&\\ +A_3[(1-a_1i)x-a_2ix^2-a_3ix^3-\ldots.]^3&\\ +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\end{aligned}$$

---

* In this and all of the following work the developments are all made about the origin, as no generality is lost and much simplification results.

Putting $1+a_1i=a$, $1-a_1i=\bar{a}$, the following equalities are obtained upon equating coefficients of like powers of x:

$$\left.\begin{aligned} &A_1\bar{a}=a,\\ -&A_1a_2i+A_2\bar{a}^2=a_2i,\\ -&A_1a_3i-2A_2\bar{a}a_2i+A_3\bar{a}^3=a_3i,\\ -&A_1a_4i-A_2(2\bar{a}a_3i+a_2^2)-3A_3\bar{a}^2a_2i+A_4\bar{a}^4=a_4i,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \qquad (2)$$

Or, solving for A_1, A_2, A_3, A_4, ..., we have

$$\begin{aligned} A_1&=\frac{a}{\bar{a}},\\ A_2&=\frac{2a_2i}{\bar{a}^3},\\ A_3&=\frac{-4a_2^2+2\bar{a}a_3i}{\bar{a}^5},\\ A_4&=\frac{-10\bar{a}a_2a_3+2(\bar{a}^2a_4-5a_2^3)i}{\bar{a}^7},\\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

Now, let $x=b_1y+b_2y^2+\dots.$ be the inverse function of $y=\bar{a}x-ia_2x^2-ia_3x^3-\dots..$ Then $x=b_1(\bar{a}x-ia_2x^2-\dots.)+b_2(\bar{a}x-ia_2x^2-\dots.)+\dots.$ is an identity and upon equating coefficients of like powers of x the following equations result:

$$\begin{aligned} &b_1\bar{a}=1,\\ -&ib_1a_2+b_2C_1^2=0,\\ -&ib_1a_3+b_2C_2^2+b_3C_1^3=0,\\ &\dots\dots\dots\dots\dots\dots\dots\dots,\\ -&ib_1a_n+b_2C_{n-1}^2+b_3C_{n-2}^3+\dots.+b_nC_1^n=0,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{aligned}$$

where the symbol C_{n+1}^m denotes the coefficient of x^{m+n} in the expansion of $(\bar{a}x-ia_2x^2-ia_3x^3-\dots.)^m$, $n>0$, $m>1$, and C_1^m denotes $\bar{a}^m$. Solving this system of equations for b_n, we obtain for $n>1$

$$b_n=\frac{(-1)^{n-1}}{\bar{a}^{\frac{n}{2}(n+1)}}\begin{vmatrix} \bar{a} & 0 & 0 & \dots. & 0 & 1\\ -ia_2 & C_1^2 & 0 & \dots. & 0 & 0\\ -ia_3 & C_2^2 & C_1^3 & \dots. & 0 & 0\\ -ia_4 & C_3^2 & C_2^3 & \dots. & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ -ia_n & C_{n-1}^2 & C_{n-2}^3 & \dots. & C_2^{n-1} & 0 \end{vmatrix}.$$

Likewise, on solving the system of equations (2) for A_n, we obtain for $n>1$

$$A_n=\frac{(-1)^{n-1}}{\bar{a}^{\frac{n}{2}(n+1)}}\begin{vmatrix} \bar{a} & 0 & 0 & \dots & 0 & a \\ -ia_2 & C_1^2 & 0 & \dots & 0 & ia_2 \\ -ia_3 & C_2^2 & C_1^3 & \dots & 0 & ia_3 \\ -ia_4 & C_3^2 & C_2^3 & \dots & 0 & ia_4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -ia_n & C_{n-1}^2 & C_{n-2}^3 & \dots & C_2^{n-1} & ia_n \end{vmatrix}.$$

It is readily seen that $A_n=2b_n$, $n>1$. Consequently, we have the

THEOREM: *The symmetry determined by the analytic arc C: $y=a_1x+a_2x^2+\dots$ is given by*

$$Z=2w-\bar{z},$$

where $z=w-i(a_1w+a_2w^2+\dots)$.

It is obvious that an analytic arc C which intersects the analytic arc C_1 which determines the symmetry S_1 is transformed by the symmetry S_1 into another analytic arc C' such that the magnitude of the angle between the arcs C and C_1 is equal to the magnitude of the angle between the arcs C_1 and C'. C' is said to be the symmetric image of C with respect to the analytic arc C_1. Obviously, C is also the symmetric image of C' with respect to the analytic arc C_1.

As an application of the above formula, we prove the following

THEOREM: *If C_1' is the symmetric image of the analytic arc C_1 with respect to the analytic arc C, C_2' the symmetric image of the analytic arc C_2 with respect to arc C, and arc C_2 the symmetric image of the analytic arc C with respect to the analytic arc C_1, then the arc C_2' is the symmetric image of the arc C with respect to the arc C_1'.*

Without loss of generality we take the arc C as the segment of the x-axis in the neighborhood of the origin; for the above configuration may be conformally transformed so that the arc C is transformed into the segment of the x-axis in the vicinity of the origin. Let the arc C_2 be given by $y=a_1x+a_2x^2+\dots$; then the arc C_2' is given by $y=-(a_1x_1+a_2x^2+\dots)$. Any point $(x, 0)$ on C in the vicinity of the origin is transformed into $Z_1=2w-x$, where $x=w-i(a_1w+a_2w^2+\dots)$ by the symmetry determined by the arc C_2. This point is transformed into $Z_2=2\bar{w}-x$ by the reflection with respect to the x-axis and is, of course, on the arc C_1'. But this latter point is the point which corresponds to the point $(x, 0)$ under the symmetry determined by the arc C_2'; for if we call this point Z_2', we have $Z_2'=2w_1-x$, where $x=w_1+i(a_1w_1+a_2w_1^2+\dots)$. Since $\bar{w}=w_1$, we have $Z_2=Z_2'$, whence the theorem.

Another such theorem is the following:

THEOREM: *If the arc C_2 is the symmetric image of the analytic arc C with respect to the analytic arc C_1, the arc C_3 the symmetric image of the arc C_1 with respect to the arc C_2, and the arc C_4 the symmetric image of the arc C_2 with respect to the arc C_3, then the arc C_4 is the symmetric image of the arc C with respect to the arc C_2.*

§ 2. *In the Plane of Complex Points.*

In the extended complex plane of analysis the coordinates x, y of the point (x, y) may be real or complex numbers. In that plane the general non-singular conformal transformation which leaves the origin fixed, and which is defined in the neighborhood of the origin, is given by

$$U=f(u)=\alpha_1 u+\alpha_2 u^2+\ldots, \qquad \alpha_1\neq 0,$$
$$V=g(v)=\beta_1 v+\beta_2 v^2+\ldots, \qquad \beta_1\neq 0,$$

where the coefficients are real or complex and where $u=x+iy$ and $v=x-iy$. If β_i is the conjugate of α_i, the transformation is then a real non-singular conformal transformation which, however, is defined for complex values of the coordinates of a point as well as for real values.

Definition. In the complex plane a complex analytic arc is the set of points (x, y), $[(x, y)]$, whose coordinates satisfy the equation $y=y_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots$ or $x=x_0+a_1(y-y_0)+a_2(y-y_0)^2+\ldots$, where the power series has a radius of convergence different from zero and the coefficients are real or complex. If all the coefficients are real, the set of points which satisfy the above equation is said to be a real analytic arc in the complex plane. Unless the contrary is stated, an analytic arc will hereafter mean a complex analytic arc.

The definition given above of two real points which are symmetric with respect to a real analytic arc is extended to apply to the complex plane in that the arc C may be a complex analytic arc and the transformation T a general non-singular conformal transformation. In all that follows, it is assumed that the slope of the arc at the point about which the power series which represents the arc is developed is not minimal. It may now readily be shown that two points which are symmetric with respect to an analytic arc are such that the two minimal lines on each point intersect on the analytic arc and, conversely, two points which are such that the two minimal lines on each intersect on the analytic arc are symmetric with respect to the analytic arc.* From this fact, and from the theorem that under any general non-singular conformal transformation minimal lines are transformed into minimal lines, it follows that two points which respectively correspond under the non-singular conformal transformation T to two points which are symmetric with respect to an analytic arc C are symmetric with respect to the analytic arc into which the arc C is transformed by the transformation T.

---

* This fact, as was pointed out by Kasner, may be taken as the definition of two points symmetric with respect to an analytic arc.

§ 3. *The (u, v)-plane.*

Now, given the complex plane, we may consider the set of points (u, v), $[(u, v)]$, such that $u=x+iy$ and $v=x-iy$. This set of points is called the *(u, v)-plane.* It is found more convenient to work in this plane and then to interpret the results so obtained to apply to the complex plane by means of the transformation

$$u=x+iy,$$
$$v=x-iy.$$

In the (u, v)-plane the transformation, called a pseudo-conformal transformation, which corresponds to the general non-singular conformal transformation in the complex plane defined, say, in the neighborhood of the origin, is simply given by two convergent power series in u and v respectively; if we assume that the origin is invariant, as we may for our purpose without loss of generality, the transformation is given by

$$U=\alpha_1 u+\alpha_2 u^2+\dots, \qquad \alpha_1\neq 0,$$
$$V=\beta_1 v+\beta_2 v^2+\dots, \qquad \beta_1\neq 0.$$

Definition. An analytic arc in the (u, v)-plane is the set of points (u, v), $[(u, v)]$, which satisfy the equation $v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\dots$, or $u=u_0+a_1(v-v_0)+a_2(v-v_0)^2+\dots$, the power series having a radius of convergence different from zero and the a_i being real or complex.

Obviously an analytic arc in the (u, v)-plane corresponds to an analytic arc in the complex plane and conversely. Minimal lines in the complex plane correspond to lines parallel to the coordinate axes in the (u, v)-plane. Hence:

Definition. Two points in the (u, v)-plane are said to be symmetric with respect to an analytic arc if the two lines on each which are parallel to the coordinate axes intersect on the analytic arc.

For these points correspond to two points in the complex plane which are symmetric with respect to the analytic arc which corresponds to the given analytic arc, and conversely. The equations for the symmetric transformation determined by a given analytic arc are readily derived and are of simple form.

Let (u, v) and (U, V) be any two points which are symmetric with respect to the analytic arc C: $v=a_1u+a_2u^2+\ldots.$, $a_1\neq 0$. Let the lines on the point (u, v) parallel to the u- and v-axes intersect the arc C in the points (u_1, v_1) and (u_2, v_2) respectively, and let the lines on the point (U, V) parallel to the u- and v-axes intersect the arc C in the points (u_2, v_2) and (u_1, v_1) respectively, where $u_1=U$, $v_1=v$, $u_2=u$, and $v_2=V$. But

$$v_1=a_1u_1+a_2u_1^2+\ldots.,$$
$$v_2=a_1u_2+a_2u_2^2+\ldots.,$$

whence

$$\left.\begin{aligned} v&=a_1U+a_2U^2+\ldots.,\\ V&=a_1\,u+a_2\,u^2+\ldots.,\end{aligned}\right\} \tag{10}$$

or

$$\left.\begin{aligned} U&=\alpha_1v+\alpha_2v^2+\ldots.,\\ V&=a_1u+a_2v^2+\ldots.,\end{aligned}\right\} \tag{11}$$

where α_i, $i=1, 2, \ldots.$, are the coefficients of the power series which is the inverse of the power series with the coefficients a_i, $i=1, 2, \ldots.$, and where $a_1\neq 0$. When the analytic arc in the complex plane which corresponds to the arc C is a real analytic arc, then the α_i, $i=1, 2, \ldots.$, in the above transformation are also the conjugates of the a_i, $i=1, 2, \ldots.$.

The problem of finding the bisector of two intersecting analytic arcs may again be simplified somewhat by considering the normal form in which one analytic arc is taken as a rectilinear segment. In the (u, v)-plane the rectilinear segment will be taken as the segment of the line $v=u$ in the neighborhood of the origin, as this line corresponds to the line $y=0$ in the complex plane.

Let, then, $v=f(u)=a_1u+a_2u^2+\ldots.$, $a_1\neq 0$, be any analytic arc, taken as passing through the origin. Let the symmetric image of the segment of the line $v=u$ in the neighborhood of the origin with respect to the analytic arc C: $v=f(u)$ be $v=g(u)=b_1u+b_2u^2+\ldots.$. Then by aid of the transformation (10) or (11) we obtain the equation

$$f[f(u)]=g(u). \tag{12}$$

Hence, the problem of finding the bisector of two intersecting analytic arcs is equivalent to solving the functional equation (12) for the unknown function $f(u)$ which must be analytic in the neighborhood of the origin, $g(u)$ being a given function of u, analytic in the vicinity of the origin, and the coefficient of its first-degree term not vanishing.

§ 4. *Conformal Equivalence of a Pair of Intersecting Arcs to a Pair of Rectilinear Segments.*

Obviously, any two intersecting analytic arcs which are conformally equivalent to two intersecting rectilinear segments must have two bisectors which are the conformal transforms of the rectilinear bisectors of these two rectilinear segments. For a large class of configurations consisting of two intersecting analytic arcs, a non-singular conformal transformation may be found which transforms any one of these configurations into two intersecting rectilinear segments.

Consider, now, the normal form of a configuration consisting of two intersecting analytic arcs, the arcs

$$\begin{aligned} C_1: &\quad V=U, \\ C_2: &\quad V=A_1U+A_2U^2+\ldots., \qquad A_1\neq 0. \end{aligned}$$

Then, there exists an infinite number of non-singular pseudo-conformal transformations which transform the above configuration into the two intersecting rectilinear segments in the neighborhood of the origin of the lines

$$\begin{aligned} C_3: &\quad v=u, \\ C_4: &\quad v=A_1u, \end{aligned}$$

such that the arc C_1 is transformed into the arc C_3 and the arc C_2 into the arc C_4, provided $|A_1|\neq 1$.

Any of the required transformations must be of the form

$$\left.\begin{aligned} U=c_1'u+c_2'u^2+\ldots., &\qquad c_1'\neq 0, \\ V=d_1'v+d_2'v^2+\ldots., &\qquad d_1'\neq 0, \end{aligned}\right\} \tag{13}$$

where the power series have radii of convergence different from zero. It will be more convenient to consider the transformation which is the inverse of the transformation (13) and which, therefore, transforms the arc C_3 into the arc C_1 and the arc C_4 into the arc C_2. Let this transformation be

$$\begin{aligned} U=c_1u+c_2u^2+\ldots., &\qquad c_1\neq 0, \\ V=d_1v+d_2v^2+\ldots., &\qquad d_1\neq 0. \end{aligned}$$

Since the segment of the line $v=u$ in the vicinity of the origin is transformed into itself, we have identically

$$c_1u+c_2u^2+\ldots.=d_1u+d_2u^2+\ldots.,$$

i. e.,

$$c_n=d_n, \qquad n=1, 2, \ldots..$$

Thus the required transformations are of the form

$$\left.\begin{aligned} U=c_1u+c_2u^2+\ldots., &\qquad c_1\neq 0, \\ V=c_1u+c_2u^2+\ldots.. & \end{aligned}\right\}. \tag{14}$$

Since the rectilinear segment in the neighborhood of the origin of the line $v=A_1u$ is transformed into the analytic arc $V=A_1U+A_2U^2+\ldots.$ by the transformation (14), the following identity exists:

$$\begin{aligned} c_1A_1u+c_2A_1^2u^2+c_3A_1^3u^3+\ldots. \\ =A_1(c_1u+c_2u^2+c_3u^3+\ldots.) \\ +A_2(c_1u+c_2u^2+c_3u^3+\ldots.)^2 \\ +A_3(c_1u+c_2u^2+c_3u^3+\ldots.)^3 \\ +\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned} \tag{15}$$

whence the following set of equalities is obtained on equating coefficients of like powers of the argument:

$$\begin{aligned} &c_1A_1=c_1A_1, \\ &c_2A_1^2=c_2A_1+c_1^2A_2, \\ &c_3A_1^3=c_3A_1+2c_1c_2A_2+c_1^3A_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

or c_1 is arbitrary and

$$\left.\begin{aligned} &(A_1^2-A_1)c_2=c_1^2A_2, \\ &(A_1^3-A_1)c_3=2c_1c_2A_2+c_1^3A_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ &(A_1^n-A_1)c_n=G_n(c_1, c_2, \ldots., c_{n-1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}\right\} \tag{16}$$

where $G_n(c_1, c_2, \ldots., c_{n-1})$ is a linear function of the A_i, $i=2, 3, \ldots.$, and the coefficients of A_i are integral functions of c_i, $i=1, 2, \ldots.$; further, only the plus sign appears. The c_i may, therefore, be found formally, but it remains to prove the convergence of the resulting power series.

Let m be a positive number for which the power series $A_1U+A_2U^2+\ldots.$ converges, and let g be an upper bound of the moduli of the terms of this series when $U=m$. Then

$$|A_n| \leqq \frac{g}{m^n}.$$

Let

$$\begin{aligned} &D_1=\frac{g}{m}, \\ &D_2=\frac{1}{|A_1^2-A_1|}\frac{g}{m}D_1^2, \\ &D_3=\frac{1}{|A_1^3-A_1|}g\left(\frac{2D_1D_2}{m^2}+\frac{D_1^3}{m^3}\right), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &D_n=\frac{1}{|A_1^n-A_1|}H_n(D_1, D_2, \ldots., D_{n-1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

where $H_n(D_1, D_2, \ldots, D_{n-1})$ denotes the value of $G_n(c_1, c_2, \ldots, c_{n-1})$ when $\frac{g}{m^n}$ and D_n are substituted respectively for A_n and c_n.

Obviously each D_n is positive, and it may be shown that

$$|c_n| \leqq D_n.$$

Let $\Gamma_n(c_1, c_2, \ldots, c_{n-1})$ denote the result obtained when A_n is replaced by its absolute value in $G_n(c_1, c_2, \ldots, c_{n-1})$. Then

$$\left.\begin{aligned} |c_n| &= \frac{1}{|A_1^n - A_1|}\,|G_n(c_1, c_2, \ldots, c_{n-1})| \\ &\leqq \frac{1}{|A_1^n - A_1|}\,\Gamma(|c_1|, |c_2|, \ldots, |c_{n-1}|) \\ &\leqq \frac{1}{|A_1^n - A_1|}\,H_n(|c_1|, |c_2|, \ldots, |c_{n-1}|), \end{aligned}\right\} \tag{α}$$

since $|A_n| \leqq \frac{g}{m^n}$.

Now take c_1, such that

$$|c_1| \leqq \frac{g}{m} = D_1;$$

then

$$|c_2| \leqq \frac{1}{|A_1^2 - A_1|}\,\frac{g}{m}\,D_1^2,$$

and by complete induction and the above inequality (α) it follows that

$$|c_n| \leqq \frac{1}{|A_1^n - A_1|}\,H_n(D_1, D_2, \ldots, D_{n-1}) = D_n.$$

Further, from the inequality

$$\big|\,|A_1|^n - |A_1|\,\big| \leqq |A_1^n - A_1|$$

it follows that for p sufficiently large, $r > p$,

$$|A_1^r - A_1| > 1 \text{ if } |A_1| > 1, \quad \text{and} \quad |A_1^r - A_1| > \frac{|A_1|}{2} \text{ if } |A_1| < 1 \text{ but } \neq 0.$$

Consequently, the set of numbers $|A_1^n - A_1|$, $n = 2, 3, 4, \ldots$, has a lower bound which is greater than zero. Let b be this lower bound. Put

$$\left.\begin{aligned} D_1' &= \frac{g}{m}, \\ D_2' &= \frac{g}{bm^2} D_1'^2, \\ D_3' &= \frac{g}{b}\left(\frac{2D_1'D_2'}{m^2} + \frac{D_1'^3}{m^3}\right), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ D_n' &= \frac{1}{b} H_n(D_1', D_2', \ldots, D^1_{n-1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{17}$$

Obviously, $D_n' \geqq D_n \geqq |c_n|$.

Further, consider the equation

$$y=\frac{g}{m}x+\frac{g}{b}\left(\frac{y^2}{m^2}+\frac{y^3}{m^3}+\frac{y^4}{m^4}+\ldots\right). \tag{18}$$

The right-hand member is convergent for all values of x and for values of y such that $|y|<m$; and, further, the coefficient of the first-degree term in y appearing in the equation is not zero and there is no absolute term. Consequently, there exists a unique solution of this equation giving y explicitly in terms of x as a power series convergent in some circle about the origin with a non-vanishing radius and vanishing there. The solution is

$$y=D_1'x+D_2'x^2+D_3'x^3+\ldots;$$

for by comparing the original identity (15) and the derived set of equalities (16) and (17) with the equation (18) it will be seen that

$$\begin{aligned}bD_1'x+bD_2'x^2+bD_3'x^3+bD_4'x^4+\ldots&\\ =bD_1'x+\frac{g}{m^2}(D_1'x+D_2'x^2+\ldots)^2&\\ +\frac{g}{m^3}(D_1'x+D_2'x^2+\ldots)^3&\\ +\ldots\ldots\ldots\ldots\ldots\ldots&\end{aligned}$$

is a formal identity.

Since $|c_n|\leqq D_n'$, it follows that the series $y=c_1x+c_2x^2+\ldots$ has a circle of convergence with a radius different from zero. Consequently, the transformation found is convergent.*

When the analytic arc C_2 corresponds to a real analytic arc in the real or complex plane, the modulus of A_1 is unity. In this case, no transformation as the above may exist, even formally. However, when the angle between the two real analytic arcs is incommensurable with π, which implies that $|A_1^n-A_1|\neq 0$, an infinite number of formal solutions exist. The question of convergence of the formal solutions obtained when $|A_1|=1$ and $A_1^n\neq 1$ for all positive integral values of n is not settled.† If $A_1^n=1$, where n is the smallest such positive integer and >1, then in the formal solution the coefficients of $c_{n+1}, c_{2n+1}, \ldots, c_{mn+1}, \ldots$ vanish. If c_{mn+1}, $m=1, 2, \ldots$, are taken equal to zero, there exists an infinite set of polynomials of the A_i, obtained from the set of equations (16), which must vanish in order that a formal solu-

---

*This result is contained in the work by G. Koenigs, "Nouvelles recherches sur les équations fonctionnelles," *Annales de l'École Normale,* 1884. However, the above proof is much more direct and more simple.

† Following a suggestion of Professor Birkhoff, I have shown that in this case, for certain arcs $V=A_1U+A_2U^2+\ldots$, the formal solutions referred to are divergent. This proof will appear in a later paper.

tion be possible. Suppose that all these polynomials do vanish. The c's thus determined, c_{mn+1}, $m=1, 2, \ldots.$, being taken to equal zero, are the coefficients of a convergent power series, for the above proof may be easily modified to apply in this case, *i. e.*, the D_{mn+1}, $m=1, 2, \ldots.$, may be taken equal to zero and b may be taken as the smallest of the $|A_1^m - A_1|$, $m=2, 3, \ldots., n$. Thus, in this case, the given configuration is pseudo-conformally equivalent to a pair of intersecting rectilinear segments. Conversely, if the given configuration is pseudo-conformally equivalent to a pair of intersecting rectilinear segments, then the equalities involving the A_i referred to above exist, for it may be shown that if the non-singular pseudo-conformal transformation T transforms the pair of intersecting rectilinear segments $v=A_1u$, $v=u$, in the vicinity of the origin, into the given configuration, then there exists a non-singular pseudo-conformal transformation T' which also transforms the pair of rectilinear segments into the given configuration, such that the coefficients of u^{mn+1} and v^{mn+1}, $m=1, 2, \ldots.$, vanish.

Let T be given by

$$U=c_1u+c_2u^2+\ldots.,$$
$$V=c_1v+c_2v^2+\ldots., \qquad c_1 \neq 0,$$

and T' by

$$U=\gamma_1u+\gamma_2u^2+\ldots.,$$
$$V=\gamma_1v+\gamma_2v^2+\ldots., \qquad \gamma_1 \neq 0,\ \gamma_{mn+1}=0,\ m=1, 2, \ldots..$$

Then a transformation T_1, given by

$$U=d_1u+d_{n+1}u^{n+1}+d_{2n+1}u^{2n+1}+\ldots.,$$
$$V=d_1v+d_{n+1}v^{n+1}+d_{2n+1}v^{2n+1}+\ldots., \qquad d_1 \neq 0,$$

may be found such that $T_1T=T'$. The transformation T_1 leaves the pair of rectilinear segments invariant, and consequently T', the product of T_1 and T, transforms the pair of rectilinear segments into the given configuration. In order that this be so, it is sufficient that we have identically

$$c_1(d_1u+d_{n+1}u^{n+1}+\ldots.)+c_2(d_1u+d_{n+1}u^{n+1}+\ldots.)^2+\ldots.=\gamma_1u+\gamma_2u^2+\ldots..$$

On equating coefficients of u^{mn+1}, $m=0, 1, 2, \ldots.$, we have

$$c_1d_1=\gamma_1,$$
$$c_1d_{n+1}+c_{n+1}d_1^{n+1}=0,$$
$$c_1d_{2n+1}+(n+1)c_{n+1}d_1^nd_{n+1}+c_{2n+1}d_1^{2n+1}=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$c_1d_{mn+1}+G(d_1, \ldots., d_{(m-1)n+1}, c_1, \ldots., c_{mn+1})=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

whence the d's are determined uniquely for γ_1 given.

Consider the equation $c_1y+c_{n+1}y^{n+1}+c_{2n+1}y^{2n+1}+\ldots\ldots=\gamma_1x$. The left-hand member is convergent in a circle about the origin with a non-vanishing radius, and since $c_1\neq 0$ and since there is no absolute term, there exists a unique solution, $y=f(x)$, which is analytic about the origin and vanishes there. This solution is precisely the power series $y=d_1x+d_{n+1}x^{n+1}+\ldots\ldots$ Consequently, the existence of the transformation T_1 is demonstrated, and hence that of T'.

Such a set of polynomials of the A_i as obtained above may also be gotten as follows: Let the configuration consisting of the analytic arc $V=A_1U+A_2U^2+\ldots$, $A_1^n=1$, n being the smallest such positive integer, and the segment of the line $V=U$ in the vicinity of the origin be pseudo-conformally equivalent to a pair of intersecting rectilinear segments; then we have the identity, in symbolic form, $cA_1=Ac$, where A_1 denotes the function $v=A_1u$, A the function $v=A_1u+A_2u^2+\ldots$ and c the function $U=c_1u+c_2u^2+\ldots\ldots$ It then follows that $cA_1c^{-1}=A$, where c^{-1} denotes the inverse function of $U=c_1u+c_2u^2+\ldots$, and consequently $(cA_1c^{-1})^n=1=A^n$, since $A_1^n=1$, where F^n denotes the n-th iteration of the function F. From the latter identity an infinite number of polynomials of the A_i is obtained and the vanishing of each is necessary for the pseudo-conformal equivalence of the given configuration to a pair of intersecting rectilinear segments. Conversely, if these conditions are satisfied, that is, if $A^n=1$, then the given configuration is pseudo-conformally equivalent to a pair of intersecting rectilinear segments. Suppose that the given configuration is not pseudo-conformally equivalent to a pair of intersecting rectilinear segments. Then the non-singular pseudo-conformal transformation, given by

$$\begin{aligned} U&=d_1u+d_2u^2+\ldots, \\ V&=d_1v+d_2v^2+\ldots, \qquad d_1\neq 0, \end{aligned}$$

which is the inverse of a transformation of which the first rn coefficients, with the exception of the first one, are determined by the first $rn-1$ equations of the set (16) when c_{mn+1}, $m=1, 2, \ldots, r$, are taken equal to zero, and the remaining coefficients arbitrary, transforms the segment of the line $V=U$ in the vicinity of the origin into itself and the analytic arc $V=A_1U+A_2U^2+\ldots$ into the analytic arc $V=A_1U+B_{rn+1}U^{rn+1}+B_{rn+2}U^{rn+2}+\ldots$, where $B_{rn+1}\neq 0$. (The converse of this is obvious; that is, a configuration consisting of the analytic arc $V=B_1U+B_{rn+1}U^{rn+1}+\ldots$, $B^n=1$ and $B_{rn+1}\neq 0$, and the segment of the line $V=U$ in the vicinity of the origin, is not pseudo-conformally equivalent to a pair of intersecting rectilinear segments; for if this were so, then we would have the identity

$$B_1(c_1u+c_2u^2+\ldots)+B_{rn+1}(c_1u+c_2u^2+\ldots)^{rn+1}+\ldots=c_1B_1u+c_2B_1^2u^2+\ldots$$

or $c_{rn+1}B_1+c_1^{rn+1}B_{rn+1}=c_{rn+1}B_1^{rn+1}$, and since $B_1^n=1$ and $c_1\neq 0$, B_{rn+1} must vanish, which is contradictory. From the fact just proved follows readily another proof of the necessity of the conditions referred to on p. 409.) We have now, symbolically, the identity $dA=Bd$, where B denotes the function $V=A_1U+B_{rn+1}U^{rn+1}+\dots$ and d the function $U=d_1u+d_2u^2+\dots$. Suppose that $A^n=1$; then we have the identity $(dAd^{-1})^n=1=B^n$. However, B^n is given by $V=U+nB_1^{n-1}B_{rn+1}U^{rn+1}+\dots$, and since $n>1$ and $B_{rn+1}\neq 0$, B^n cannot be identity. Consequently, if $A^n=1$, the given configuration is pseudo-conformally equivalent to a pair of intersecting rectilinear segments. In particular, we may state the

THEOREM: *A necessary and sufficient condition that a configuration consisting of two real analytic arcs which intersect at an angle which is commensurable with π,* but which are not tangent at the point of intersection, be conformally equivalent to a pair of intersecting rectilinear segments, is the vanishing of each of an infinite number of rational functions, determined as indicated above, of the coefficients of the equations of the arcs.*

Let C_1: $v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\dots$, $a_1\neq 0$, and C_2: $v=v_0+b_1(u-u_0)+b_2(u-u_0)^2+\dots$, $b_1\neq 0$, be two analytic arcs intersecting in the point (u_0, v_0). If the arc C_1 is transformed into the analytic arc C_1': $v=\alpha_1u+\alpha_2u^2+\dots$ and the arc C_2 into the analytic arc C_1': $v=\beta_1u+\beta_2u^2+\dots$ by the translation

$$U=u-u_0,$$
$$V=v-v_0,$$

then $a_1=\alpha_1$, $b_1=\beta_1$. If the arcs C_1', C_2' are transformed by a non-singular pseudo-conformal transformation into the analytic arc $v=A_1u+A_2u^2+\dots$ and the segment in the vicinity of the origin of the line $v=u$, respectively, then $\frac{\alpha_1}{\beta_1}=A_1$, and conversely. Therefore $\left|\frac{a_1}{b_1}\right|=\left|\frac{\alpha_1}{\beta_1}\right|=|A_1|$. We may now state the

THEOREM: *The two intersecting analytic arcs*

$$C_1: \quad v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\dots, \qquad a_1\neq 0,$$
$$C_2: \quad v=v_0+b_1(u-u_0)+b_2(u-u_0)^2+\dots, \qquad b_1\neq 0,$$

where $\left|\frac{a_1}{b_1}\right|\neq 1$, *are pseudo-conformally equivalent to a pair of intersecting rectilinear segments.*

---

* This corresponds to the above condition that $A^n=1$.

We also have the

THEOREM: *The two intersecting analytic arcs*

$$\begin{aligned} C_1: &\quad v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\ldots, &\quad a_1\neq 0,\\ C_2: &\quad v=v_0+b_1(u-u_0)+b_2(u-u_0)^2+\ldots, &\quad b_1\neq 0, \end{aligned}$$

where $\left|\frac{a_1}{b_1}\right|\neq 1$ *have just two symmetric bisectors.*

That there are just two bisectors follows from the fact that there are just two formal solutions for the coefficients of the power series which represents the bisector, as will be seen below.

We have the two corresponding theorems in the complex plane when the corresponding condition on the two analytic arcs, using the same notation as above, $\left|\frac{1+a_1b_1-(a_1-b_1)i}{1+a_1b_1+(a_1+b_1)i}\right|\neq 1$ is verified. Further, from the above developments we have the

THEOREM: *The functional equation*

$$f[f(u)]=g(u),$$

where $g(u)$ *is an analytic function defined in the vicinity of the origin and vanishing there has just two solutions,* $f(u)$, *which are analytic in the vicinity of the origin and which vanish there also, provided the modulus of the first-degree term of* $g(u)\neq 0, 1$.

Another result* also follows from the above developments. It is the

THEOREM: *Let* S *and* S' *be any two real non-singular pseudo-conformal transformations, defined in the vicinity of the point* $z_0=x_0+iy_0$, *which is left invariant by both, such that the coefficients of* $(z-z_0)$ *of each transformation, given as a development in powers of* $(z-z_0)$, *are equal and have a modulus not equal to unity. Then* S *and* S' *are equivalent in the sense that* S *may be transformed into* S' *by a non-singular pseudo-conformal transformation.*

Let S and S' be given by

$$\begin{aligned} S\ : &\quad Z=z_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots,\\ S': &\quad Z=z_0+a_1(z-z_0)+b_2(z-z_0)^2+\ldots, &\quad z=x+iy,\ a_1\neq 0,\ |a_1|\neq 1. \end{aligned}$$

In order that

$$T:\quad Z=z_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots,\qquad c_1\neq 0,$$

be a transformation such that $T^{-1}ST=S'$ or $ST=TS'$, it is necessary and sufficient that we have identically

---

* This interpretation was suggested by Professor Kasner.

$$
\begin{aligned}
z_0 &+ c_1(a_1\xi + a_2\xi^2 + \dots\dots) \\
&+ c_2(a_1\xi + a_2\xi^2 + \dots\dots)^2 \\
&+ \dots\dots\dots\dots\dots\dots\dots \\
&\qquad = z_0 + a_1(c_1\xi + c_2\xi^2 + \dots\dots) \\
&\qquad\quad + b_2(c_1\xi + c_2\xi^2 + \dots\dots)^2 \\
&\qquad\quad + \dots\dots\dots\dots\dots\dots\dots,
\end{aligned}
$$

where $\xi = z - z_0$. This is the same identity which is obtained when the analytic arc $v = z_0 + a_1(u - z_0) + a_2(u - z_0)^2 + \dots\dots$ is transformed pseudo-conformally into the analytic arc $v = z_0 + a_1(u - z_0) + b_2(u - z_0)^2 + \dots\dots$ and the segment of the line $v = u$ in the vicinity of the point (z_0, z_0) into itself. Thus, by the theorem at the bottom of p. 409 concerning the pseudo-conformal equivalence of a pair of intersecting analytic arcs to a pair of intersecting rectilinear segments, the existence of the transformation T follows.

§ 5. *Direct Discussion of the Bisection Problem.*

As stated above, the problem of finding a bisector of two intersecting analytic arcs is equivalent to solving the functional equation

$$a_1(a_1u + a_2u^2 + \dots\dots) + a_2(a_1u + a_2u^2 + \dots\dots)^2 + \dots\dots = b_1u + b_2u^2 + \dots\dots,$$

where $b_1u + b_2u^2 + \dots\dots$ is the given function, analytic about the origin and $b_1 \neq 0$, and $a_1u + a_2u^2 + \dots\dots$ the unknown function, also required to be analytic about the origin. Upon equating coefficients of like powers of the argument, the following set of equalities is obtained:

$$
\begin{aligned}
a_1^2 &= b_1, \\
a_1a_2 + a_2a_1^2 &= b_2, \\
a_1a_3 + 2a_1a_2^2 + a_3a_1^3 &= b_3, \\
\dots\dots\dots\dots\dots&\dots,
\end{aligned}
$$

or

$$
\left.
\begin{aligned}
a_1^2 &= b_1, \\
(a_1 + a_1^2)\,a_2 &= b_2, \\
(a_1 + a_1^3)\,a_3 + 2a_1a_2^2 &= b_3, \\
\dots\dots\dots\dots\dots\dots&\dots, \\
(a_1 + a_1^m)\,a_m + P(a_1, a_2, \dots\dots, a_{m-1}) &= b_m, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots&\dots,
\end{aligned}
\right\} \qquad (19)
$$

where $P_m(a_1, a_2, \dots\dots, a_{m-1})$ is a polynomial in a_i, $i = 1, 2, \dots\dots, (m-1)$. If $a_1^m \neq -1$ for all positive integral values of m, a_i, $i = 2, 3, \dots\dots$, may be found formally, and hence if $b_1^m \neq 1$ for all positive integral values, there exist two formal solutions which, if $|b_1| \neq 1$, are both convergent in a circle of radius different from zero; if $|b_1|=1$, the question of convergence, in general, of either of the two formal solutions is unsettled.*

Consider the configuration consisting of the segments of the lines $v=u$ and $v=b_1u$ in the vicinity of the origin. Let this configuration have the segment in the vicinity of the origin of the line $v=a_1u$ as a bisector, where $a_1^n=-1$, n being the smallest such positive integer. Then $b^n=1$. The given configuration may be transformed into itself by an infinitude (of potency c) of non-singular pseudo-conformal transformations; for let

$$\begin{aligned} U&=c_1u+c_2u^2+\dots\dots, \\ V&=c_1v+c_2v^2+\dots\dots, \qquad c_1\neq 0, \end{aligned}$$

be any non-singular pseudo-conformal transformation which transforms the segment in the vicinity of the origin of the line $v=u$ into itself. Then, in order that this transformation transform the segment in the vicinity of the origin of the line $v=b_1u$ into itself, it is necessary and sufficient that the following identity exist:

$$c_1b_1u+c_2(b_1u)^2+c_3(b_1u)^3+\dots\dots=b_1(c_1u+c_2u^2+c_3u^3+\dots\dots),$$

or

$$\begin{aligned} c_1b_1&=c_1b_1, \\ c_2b_1^2&=c_2b_1, \\ &\dots\dots\dots, \\ c_nb_1^n&=c_nb_1, \\ &\dots\dots\dots, \end{aligned}$$

whence c_1, c_{n+1}, c_{2n+1}, $\dots\dots$, c_{mn+1}, $\dots\dots$ are arbitrary and $c_r=0$, $r\neq mn+1$, $m=0, 1, 2, \dots\dots$.

Let the transformation

$$\begin{aligned} U&=c_1u+c_{n+1}u^{n+1}+c_{2n+1}u^{2n+1}+\dots\dots, \\ V&=c_1v+c_{n+1}v^{n+1}+c_{2n+1}v^{2n+1}+\dots\dots, \qquad c_1\neq 0, \end{aligned}$$

transform the segment in the vicinity of the origin of the line $v=a_1u$ into the analytic arc C: $V=A_1U+A_2U^2+A_3U^3+\dots\dots$; then we have identically

$$\begin{aligned} c_1(a_1u)&+c_{n+1}(a_1u)^{n+1}+c_{2n+1}(a_1u)^{2n+1}+\dots\dots \\ &=A_1(c_1u+c_{n+1}u^{n+1}+c_{2n+1}u^{2n+1}+\dots\dots) \\ &+A_2(c_1u+c_{n+1}u^{n+1}+c_{2n+1}u^{2n+1}+\dots\dots)^2 \\ &+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{aligned} \tag{20}$$

from which it follows that

---

* The second foot-note on p. 407 has reference here also.

$$\left.\begin{aligned}
&A_1=a_1,\\
&A_2=A_3=\ldots\ldots=A_n=0,\\
&c_{n+1}a_1^{n+1}=a_1c_{n+1}+A_{n+1}c_1^{n+1},\\
&A_{n+2}=A_{n+3}=\ldots\ldots=A_{2n}=0,\\
&c_{2n+1}a_1^{2n+1}=a_1c_{2n+1}+(n+1)A_{n+1}c_1^nc_{n+1}+c_1^{2n+1}A_{2n+1},\\
&A_{2n+2}=A_{2n+3}=\ldots\ldots=A_{3n}=0,\\
&c_{3n+1}a_1^{3n+1}=a_1c_{3n+1}+(n+1)A_{n+1}c_1^nc_{2n+1}+\frac{(n+1)n}{2}A_{n+1}c_1^{n-1}c_{n+1}^2\\
&\qquad\qquad+(2n+1)A_{2n+1}c_1^{2n}c_{n+1}+A_{3n+1}c_1^{3n+1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}\quad(21)$$

As on p. 408, it can readily be shown that the set of A's which satisfy (21) also satisfy a similar set of equations in which $c_{2mn+1}=0$, $m=1, 2, \ldots\ldots$ We may then assume without loss of generality that $c_{2mn+1}=0$, and we have, in general,

$$c_1^{(2m-1)n+1}A_{(2m-1)n+1}=-2a_1c_{(2m-1)n+1}+G_{(2m-1)n+1}(A_1, A_n, \ldots\ldots, A_{(2m-2)n+1}), \quad (22)$$

$$A_{2mn+1}=G_{2mn+1}(A_1, A_n, \ldots\ldots, A_{(2m-1)n+1}), \quad (23)$$

$$A_r=0; \qquad r\neq kn+1;\ k=0, 1, 2, \ldots\ldots \quad (24)$$

The equalities (23) are obviously necessary conditions that $V=A_1U+A_2U^2+\ldots\ldots$ be a bisector of the given configuration; they can also be shown to be sufficient. In other words, a non-singular pseudo-conformal transformation

$$\begin{aligned}
U&=c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots,\\
V&=c_1v+c_{n+1}v^{n+1}+c_{3n+1}v^{3n+1}+\ldots\ldots, \qquad c_1\neq 0,
\end{aligned}$$

may be found such that the given pair of intersecting rectilinear segments is transformed into itself and the arc C: $V=A_1U+A_2U^2+\ldots\ldots$ into the segment in the vicinity of the origin of the line $v=a_1u$. The proof of this is similar to the convergence proof on pp. 404–407. Corresponding to the identity (15), we have the identity

$$\begin{aligned}
&c_1a_1u+c_{n+1}(a_1u)^{n+1}+c_{3n+1}(a_1u)^{3n+1}+\ldots\ldots\\
&\quad=A_1(c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots)\\
&\quad+A_n(c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots)^n\\
&\quad+A_{2n+1}(c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots)^{2n+1}\\
&\quad+A_{3n+1}(c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots)^{3n+1}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

whence the following equalities to determine the $c_{(2m-1)n+1}$:

$$\begin{aligned}
&c_1 a_1 = A_1 c_1 ,\\
&-2A_1 c_{n+1} = c_1^{n+1} A_{n+1} ,\\
&-2A_1 c_{3n+1} = \frac{(n+1)n}{2} A_{n+1} c_1^{n-1} c_{n+1}^2 + (2n+1) c_1^{2n} A_{2n+1} + c_1^{3n+1} A_{3n+1} ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\\
&-2A_1 c_{(2m-1)n+1} = G_{(2m-1)n+1}(c_1 , c_{n+1} , \ldots , c_{(2m-3)n+1}) ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,
\end{aligned}$$

where $G_{(2m-1)n+1}(c_1 , c_{n+1} , \ldots , c_{(2m-3)n+1})$ is a linear function of the A_{kn+1} and the coefficients of the A_{kn+1} are integral functions of the c's. Corresponding to the set of equations (17), we have

$$\begin{aligned}
&D_1 = \frac{g}{\mu} ,\\
&D_{n+1} = \frac{1}{|2A_1|} \frac{g}{\mu} D_1^{n+1} ,\\
&D_{2n+1} = \frac{1}{|2A_1|} g \left[\frac{n+1}{\mu^{n+1}} D_1^n D_{n+1} + \frac{D_1^{2n+1}}{\mu^{2n+1}} \right] ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\\
&D_{mn+1} = \frac{1}{|2A_1|} H_{mn+1}[D_1 , D_{n+1} , \ldots , D_{(m-1)n+1}] ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots , \qquad m = 0, 1, 2, \ldots ,
\end{aligned}$$

where μ is a positive number for which the series $A_1 U + A_{n+1} U^{n+1} + \ldots$ converges and g is an upper bound of the moduli of the terms of the series when $U = \mu$. Corresponding to the equation (18), we have the equation

$$V = \frac{g}{\mu} U + \frac{g}{|2A_1|} \left[\frac{V^{n+1}}{\mu^{n+1}} + \frac{V^{2n+1}}{\mu^{2n+1}} + \ldots + \frac{V^{mn+1}}{\mu^{mn+1}} + \ldots \right] ,$$

the left-hand member of which is convergent for all values of U and for $|V| < \mu$.

From (22) and (23) it is seen that $A_{(2m-1)n+1}$, $m = 1, 2, \ldots, p =$ any positive integer, are arbitrary and that if $A_{(2m-1)n+1}$, $m = 1, 2, \ldots, p$, are given, then A_{2mn+1}, $m = 1, 2, \ldots, p$, are determined uniquely. The set of relations (23) is equivalent to the set of relations between the coefficients of the equation of the bisector gotten directly from (19) when $b_n = 0$, $n \neq 1$.

We may now state the two closely related theorems:

THEOREM: *The configuration consisting of the two intersecting segments* $v = u$ *and* $v = b_1 u$, *where* $b_1^n = 1$, $n =$ *a positive integer* > 1, *has an uncountable infinity of bisectors.*

THEOREM: *The functional equation* $f[f(u)] = b_1 u$, $b^n = 1$, $n =$ *smallest such positive integer* > 1, *has an uncountable infinity of solutions, analytic about the* *origin and vanishing there, all of which may be determined as above and are of the form* $V = A_1U + A_{n+1}U^{n+1} + A_{2n+1}U^{2n+1} + \dots$*; further,* $A_{(2m-1)n+1}$, $m = 1, 2, \dots, p =$ *any positive integer, are arbitrary when* $A_1^n = -1$.*

It is obvious that there is only one solution, viz., $V = A_1U$, when n is odd, which has A_1 equal to that square root of b_1 which has the largest amplitude θ, where θ is taken such that $0 < \theta \leqq 2\pi$; in all other cases $A_1^n = -1$, n being the smallest such positive integer.

Now, consider the configuration consisting of the segment in the vicinity of the origin of the line $v = u$ and the analytic arc C: $v = B_1u + B_2u^2 + \dots$, and let a bisector be given by $v = a_1u + a_2u^2 + \dots$, where $a_1^n = -1$, n being the smallest such positive integer. It may be shown that this configuration has not more than one bisector such that the coefficient of u in its equation is a definite value of a_1 if the configuration is not pseudo-conformally equivalent to a pair of intersecting rectilinear segments.

We may, without losing generality, consider the configuration as consisting of the analytic arc C: $v = b_1u + b_{rn+1}u^{rn+1} + \dots + b_{(r+j)n+1}u^{(r+j)n+1} + \dots + b_qu^q + \dots$, $b_{rn+1} \neq 0$, $q =$ any positive integer $\neq kn+1$, $k = 0, 1, 2, \dots$, and the segment in the vicinity of the origin of the line $v = u$.†

Obviously, if $r = 1$, there exists no formal solution for a bisector as required. Let r be greater than unity; then the coefficients $b_{(r+m)n+1}$, $m = 1, 2, \dots, r-1$, may be taken equal to zero as well as the coefficients b_s, $s \neq kn+1$, $k = 0, 1, 2, \dots, 2r-1, \dots, t$. For it may be shown that there exists a non-singular pseudo-conformal transformation which transforms the segment of the line $v = u$ in the vicinity of the origin into itself and the analytic arc

$$C: \qquad v = b_1u + b_{rn+1}u^{rn+1} + b_{(r+1)n+1}u^{(r+1)n+1} + \dots + b_{(r+m)n+1}u^{(r+m)n+1} + \dots$$

into the analytic arc

$$C': \qquad V = b_1U + B_{rn+1}U^{rn+1} + B_{(r+1)n+1}U^{(r+1)n+1} + \dots + B_{(r+m)n+1}U^{(r+m)n+1} + \dots,$$

such that $B_{(r+m)n+1} = 0$, $m = 1, 2, \dots, r-1$.

Let

$$U = c_1u + c_2u^2 + \dots,$$
$$V = c_1v + c_2v^2 + \dots, \qquad c_1 \neq 0,$$

be a transformation which transforms the segment of the line $v = u$ in the vicinity of the origin into itself. Then, in order that the analytic arc C be transformed into the analytic arc C' by the above transformation, it is necessary and sufficient that we have identically

---

* Leau and Lémeray have given some theorems on this functional equation (*Bull. de la Soc. Math. de France*, Vol. XXVI, 1898). More details of the nature of the solutions are given above.

† See p. 409.

$$\begin{aligned}
&b_1(c_1u+c_2u^2+\ldots\ldots)\\
&+B_{rn+1}(c_1u+c_2u^2+\ldots\ldots)^{rn+1}\\
&+B_{(r+1)n+1}(c_1u+c_2u^2+\ldots\ldots)^{(r+1)n+1}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+B_{(r+m)n+1}(c_1u+c_2u^2+\ldots\ldots)^{(r+m)n+1}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\qquad=c_1(b_1u+b_{rn+1}u^{rn+1}+b_{(r+1)n+1}u^{(r+1)n+1}+\ldots\ldots+b_{(r+m)n+1}u^{(r+m)n+1}+\ldots\ldots)\\
&\qquad+c_2(b_1u+b_{rn+1}u^{rn+1}+b_{(r+1)n+1}u^{(r+1)n+1}+\ldots\ldots+b_{(r+m)n+1}u^{(r+m)n+1}+\ldots\ldots)^2\\
&\qquad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The following equalities then result:

$$\begin{aligned}
b_1c_1&=b_1c_1,\\
b_1c_2&=b_1^2c_2,\\
&\ldots\ldots\ldots,\\
b_1c_{rn}&=b_1^{rn}c_{rn},
\end{aligned}$$

from which it follows that $c_1, c_{n+1}, c_{2n+1}, \ldots\ldots, c_{mn+1}, \ldots\ldots, c_{(r-1)n+1}$ are arbitrary $(c_1\neq 0)$, while

$$c_2=\ldots\ldots=c_n=c_{n+2}=\ldots\ldots=c_{2n}=c_{2n+2}=\ldots\ldots=c_{rn}=0, \qquad n>1.^*$$

Further,

$$\begin{aligned}
&b_1c_{rn+1}+c_1^{rn+1}B_{rn+1}=c_1b_{rn+1}+c_{rn+1}b_1^{rn+1},\\
&b_1c_{(r+1)n+1}+(rn+1)c_1^{rn}c_{n+1}B_{rn+1}+c_1^{(r+1)n+1}B_{(r+1)n+1}\\
&\qquad\qquad=c_1b_{(r+1)n+1}+(n+1)c_{n+1}b_1^nb_{rn+1}+c_{(r+1)n+1}b_1^{(r+1)n+1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&b_1c_{(r+m)n+1}+(rn+1)c_1^{rn}c_{mn+1}B_{rn+1}+\ldots\ldots+c_1^{(r+m)n+1}B_{(r+m)n+1}\\
&\qquad\qquad=c_1b_{(r+m)n+1}+\ldots\ldots+(mn+1)b_1^{mn}b_{rn+1}c_{mn+1}+b_1^{(r+m)n+1}c_{(r+m)n+1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

where the terms omitted in the last equality do not involve c_{mn+1} or $B_{(r+m)n+1}$. Also it is readily seen that $c_{(r+m)n+l}$, $l=2, 3, \ldots\ldots, n-1$, $n>1$, vanish. Solving for B_i, we find

$$c_1^{rn}B_{rn+1}=b_{rn+1}$$

and

$$\begin{aligned}
c_1^{(r+m)n+1}B_{(r+m)n+1}=(m-r)nb_{rn+1}c_{mn+1}\\
+G(c_1, c_{n+1}, \ldots\ldots, c_{(m-1)n+1}, b_{(r+1)n+1}, \ldots\ldots, b_{(r+m)n+1}).
\end{aligned}$$

Since $b_{rn+1}\neq 0$ and c_{mn+1} is arbitrary, the latter may be so taken that $B_{(r+m)n+1}$, $m=1, 2, \ldots\ldots, r-1$, vanishes. Hence the equation of the analytic arc C may be taken to be of the form

$$v=b_1u+b_{rn+1}u^{rn+1}+b_{2rn+1}u^{2rn+1}+\ldots\ldots.$$

---

[*] Of course, when $n=1$ this reduction is uncalled for.

Let $r=2p$, where p is a positive integer. Now, from the set of equations (19) the following two equations are obtained:

$$(a_1+a_1^{2pn+1})a_{2pn+1}+P_1(a_1,\ldots,a_{(2p-1)n+1})=b_{2pn+1},$$
$$(a_1+a_1^{(2p+1)n+1})a_{(2p+1)n+1}+(2p-1)na_{n+1}a_{2pn+1}+P_{n+1}(a_1,\ldots,a_{(2p-1)n+1})=0,$$

since $a_2=\ldots=a_n=0$. Similarly, if, instead of the analytic arc C: $v=b_1u+b_{2pn+1}u^{2pn+1}+\ldots$, we have the segment of the line $v=b_1u$ in the vicinity of the origin, the following equations result by using the last theorem on p. 415:

$$2a_1a'_{2pn+1}+P_1(a_1,\ldots,a_{(2p-1)n+1})=0,$$
$$(2p-1)na_{n+1}a'_{2pn+1}+P_{n+1}(a_1,\ldots,a_{(2p-1)n+1})=0.$$

From the above two pairs of equations it follows that

$$2a_1(a_{2pn+1}-a'_{2pn+1})=b_{2pn+1}\neq 0$$

and

$$(2p-1)na_{n+1}(a_{2pn+1}-a'_{2pn+1})=0,$$

whence $a_{n+1}=0$. Assuming that $a_{n+1}=\ldots=a_{(2m+1)n}=0$, we obtain likewise from (19) the equation

$$(a_1+a_1^{(2p+2m+1)n+1})a_{(2p+2m+1)n+1}+(2p-2m-1)na_{(2m+1)n+1}a_{2pn+1}$$
$$+P_{(2m+1)n+1}(a_1,\ldots,a_{(2p-1)n+1})=0,\qquad \text{where } m=1,2,\ldots,p-1,$$

the first term of the left-hand member vanishing; and in the case that the segment in the vicinity of the origin of the line $v=u$ is taken instead of the analytic arc C: $v=b_1u+b_{2pn+1}u^{2pn+1}+\ldots$, we have correspondingly the equation

$$(2p-2m-1)a_{(2m+1)n+1}a'_{2pn+1}+P_{(2m+1)n+1}(a_1,\ldots,a_{(2p-1)n+1})=0,$$

whence it follows that

$$(2p-2m-1)a_{(2m+1)n+1}(a_{2pn+1}-a'_{2pn+1})=0.$$

Therefore, $a_{(2m+1)n+1}=0$, and thus, by complete induction, it follows that $a_s=0$, $s=2,3,\ldots,2pn$.

The set of equations (19) then gives further the following equations:

$(a_1+a_1^{2pn+1})a_{2pn+1}=b_{2pn+1}\neq 0,$ (a_{2pn+1} is here uniquely determined and $\neq 0$.)
$(a_1+a_1^{2pn+2})a_{2pn+2}=0,$
$\ldots\ldots\ldots\ldots\ldots\ldots,$
$(a_1+a_1^{2pn+n})a_{2pn+n}=0,$
$(a_1+a_1^{2pn+n+1})a_{2pn+n+1}=0,$ ($a_{2pn+n+1}$ is not determined here.)
$(a_1+a_1^{2pn+n+2})a_{2pn+n+2}=0,$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$
$(a_1+a_1^{2pn+3n+1})a_{2pn+3n+1}=0,$ ($a_{2pn+3n+1}$ is not determined here.)
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$
$(a_1+a_1^{4pn})a_{4pn}=0,$
$(a_1+a_1^{4pn+1})a_{4pn+1}=b_{4pn+1}-(2pn+1)a_{2pn+1}^2,$

$$(a_1+a_1^{4pn+2})\,a_{4pn+2}=b_{4pn+2}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(a_1+a_1^{4pn+n})\,a_{4pn+n}=b_{4pn+n}\,,$$

$$(a_1+a_1^{4pn+n+1})\,a_{4pn+n+1}-na_{2pn+1}a_{2pn+n+1}=b_{4pn+n+1}\,,$$

(The first term in this equation vanishes; and since $a_{2pn+1}\neq 0$, $a_{2pn+n+1}$ is uniquely determined.)

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(a_1+a_1^{2(p+m)n+n+1})\,a_{2(p+m)n+n+1}=0,$$

$$\left\{\begin{array}{l}(a_1+a_1^{2(p+m)n+n+2})\,a_{2(p+m)n+n+2}=C_{2(p+m)n+n+2}\,,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ (a_1+a_1^{2(p+m)n+8n})\,a_{2(p+m)n+3n}=C_{2(p+m)n+3n}\,,\end{array}\right.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(a_1+a_1^{2(2p+m)n+n+1})\,a_{2(2p+m)n+n+1}-(2m+1)\,na_{2pn+1}a_{2(p+m)n+n+1}=C_{2(2p+m)n+n+1}\,,$$

(The first term here vanishes and $a_{2(p+m)n+n+1}$ is uniquely determined.)

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$m=0,\ 1,\ 2,\ \ldots\ldots,$$

where the C's do not involve any a_t, $t\geqq 2(p+m)n+n+1$, and are therefore definite. The coefficients of the a's in the bracketed set of equations are different from zero, and a_t, whose coefficient $(a_1+a_1^t)$ vanishes, is determined uniquely by the equation obtained by equating the coefficients of u^{t+2pn}. It is thus seen that the determination of the a's is unique.

If b_{2pn+1} vanishes, then it is readily seen by an argument similar to that employed on p. 418 that $b_{2pn+n+1}$ must also vanish in order that a formal solution be possible, for it is found that $b_{2pn+n+1}=\dfrac{(2p-1)n}{2}\,\dfrac{a_{n+1}}{a_1}\,b_{2pn+1}$. Hence, if the arc C: $v=b_1u+b_{rn+1}u^{rn+1}+\ldots.+b_qu^q+\ldots.$ is such that r is an even positive integer, there exists a unique formal solution, while if r is an odd positive integer there exists no formal solution with the given value of a_1. If n is even, the above applies to both values which a_1 may take for a given b_1, while if n is odd, it is evident that there exists always a unique formal solution having a_1 equal to that square root of b_1 which has the largest amplitude θ, where θ is taken such that $0<\theta\leqq 2\pi$. In particular, in the case that the given curvilinear analytic arc is tangent to the segment $v=u$ at the origin, *i. e.*, $b_1=1$ and some $b_r\neq 0$, $r>1$, there exists one formal solution having $a_1=-1$ in the case that the first b_r, $r>1$, which does not vanish is the coefficient of u^{2p+1}, $p=1$, $2,\ldots.$, and no solution in the case that the first coefficient b_r, $r>1$, which does not vanish is the coefficient of u^{2p}, $p=1, 2, \ldots..$ Of course, there exists one formal solution having $a_1=+1$ in either case. Hence, we have the related theorems:

THEOREM: *The configuration consisting of the two intersecting analytic arcs which in normal form are the segment in the vicinity of the origin of the* *line* $v=u$ *and the analytic arc* $v=b_1u+b_2u^2+\ldots$, $b_1\neq 0$, *and which, if* $b_1^n=1$, $n=$ *a positive integer, are not pseudo-conformally equivalent to a pair of intersecting rectilinear segments, has at most one bisector, given by* $v=a_1u+a_2u^2+\ldots$, *corresponding to each of the two values,* $+\sqrt{b}$, $-\sqrt{b}$, *for* a_1.

THEOREM: *The functional equation*

$f[f(u)]=b_1u+b_{mn+1}u^{mn+1}+b_{mn+2}u^{mn+2}+\ldots$, $b_1^n=1$, $b_{mn+1}\neq 0$, m, $n=1, 2, \ldots$, *has at most one solution, analytic about the origin and vanishing there, corresponding to each of the two possible values of the coefficient of the first-degree term of* $f(u)$.

Definitions. A configuration which is pseudo-conformally equivalent to the configuration consisting of the segment $v=u$ in the vicinity of the origin and the analytic arc C: $v=b_1u+b_2u^2+\ldots$, $b_1^n=1$, is called a *rational angle.* If n is the smallest such positive integer, the angle is said to be of the n-th order. If $b_1^n\neq 1$ for all positive integral values of n, the configuration is called an *irrational angle.*

Consider the configuration in the complex plane consisting of the segment $y=0$ and the real analytic arc C: $y=a_1x+a_2x^2+\ldots$. Let the configuration in the (u, v)-plane which corresponds to the above consist of the segment $v=u$ and the analytic arc $v=b_1u+b_2u^2+\ldots$. Then

$$\frac{1-a_1i}{1+a_1i}=b_1,$$

from which it follows that the angle formed by the given real analytic arc and the x-axis is commensurable with π if $b_1^n=1$, and conversely. Thus we may state the

THEOREM: *If two intersecting analytic arcs have three bisectors, then the configuration is a rational angle and is conformally equivalent to a pair of intersecting rectilinear segments.*

As noted above, the problem of the conformal equivalence of two intersecting analytic arcs to a rectilinear angle is identical with the problem of the solution of the functional equation

$$\phi[g(u)]=a\phi(u), \qquad a= \text{coefficient of first-degree term of } g(u),$$

where $g(u)$ is a given function, analytic about the origin and vanishing there, and $\phi(u)$ is unknown and required to be analytic about the origin and to vanish there. This equation is known as Schroeder's equation. We have the

THEOREM: *If the functional equation*

$$f[f(u)]=g(u),$$

where $g(u)$ *is a given function, analytic about the origin and vanishing there,* *and having a non-vanishing coefficient of its first-degree term, and* $f(u)$ *is unknown and required to be analytic about the origin and to vanish there also, has at least three solutions, then Schroeder's equation, as given above, has an infinity of solutions; in this case it is necessary that some positive integral power of the coefficient of the first-degree term of* $g(u)$ *be unity.*

We may also state the

THEOREM: *The functional equation*

$$f[f(u)]=g(u)=b_1u+b_2u^2+\ldots.,\qquad b_1\neq 0,$$

where $f(u)$ *is the unknown function which is required to be analytic about the origin and to vanish there, has at most two solutions if* $b_1^n\neq 1$ *for all positive integral values of* n*; and if* $b_1^n=1$, $n=$ *a positive integer, there must be either no solution, one solution, two solutions or an infinite number of solutions.*

§ 6. *The n-Section of a Curvilinear Angle, where* $n=2^p$.

Let the analytic arc C_2: $v=B_1u+B_2u^2+\ldots.=G(u)$ be the symmetric image of the analytic arc C_1: $v=b_1u+b_2u^2+\ldots.=g(u)$ with respect to the analytic arc C: $v=a_1u+a_2u^2+\ldots.=f(u)$. The symmetry determined by the arc C is given by

$$U=\alpha_1v+\alpha_2v^2+\ldots.=f^{-1}(v),$$
$$V=a_1u+a_2u^2+\ldots.=f(u).$$

Then we have identically

$$a_1u+a_2u^2+\ldots.=B_1(\alpha_1v+\alpha_2v^2+\ldots.)+B_2(\alpha_1v+\alpha_2v^2+\ldots.)^2+\ldots.,$$

where $u=\beta_1v+\beta_2v^2+\ldots.=g^{-1}(v)$, or, symbolically,

$$f=Gf^{-1}g$$

or

$$fg^{-1}f=G;\qquad\qquad (1')$$

and if $g(u)=u$, *i. e.*, $g=1$, (1′) becomes $ff=G$.

If $g=f_1$ and $f=f_1f_1$, then $f_1f_1f_1^{-1}f_1f_1=G$ or $f_1f_1f_1=G$, and, in general, if G represents an analytic arc C_n which is such that there exists an analytic arc C_1, given by $v=f_1(u)$, such that the image of the segment $v=u$ with respect to the arc C_1 is the arc C_2 and the image of the arc C_1 with respect to the arc C_2 is the C_3, and so on until the image of the arc C_{n-2} with respect to the arc C_{n-1} is the arc C_n, then

$$f_1^n=G.$$

If the configuration consisting of the segment $v=u$ and the analytic arc C_n is not conformally equivalent to a rectilinear angle, then the functional equation

$$\phi^2=G$$

has at most one solution, as required, corresponding to a given value of the coefficient of the first-degree term (which can only have one of two values). Consequently, if $n=2r$, $r=$ a positive integer,

$$f_1^r=\phi.$$

Obviously, the configuration consisting of the segment $v=u$ and the analytic arc C: $v=\phi(u)$ is not conformally equivalent to a rectilinear angle, and therefore if $r=2t$, $t=$ a positive integer, we have again

$$f_1^t=\psi.$$

Consequently, if $n=2^p$, f_1 is uniquely determined, if it exists at all, for a given value of the coefficient of the first-degree term.

Consider now a rational angle consisting of the analytic arcs C_0, C_1 which is not conformally equivalent to a rectilinear angle. There then exists a finite number of analytic arcs $C_2, C_3, \ldots, C_n$ through the intersection, P, of the arcs C_0, C_1, such that the arc C_n is tangent to the arc C_0 at P, *i. e.*, the arcs C_0 and C_n form a horn angle, and such that the image of the arc C_m with respect to the arc C_{m+1} is the arc C_{m+2}, $m=0, 1, \ldots, n-2$. If the set of arcs $C_1, C_2, \ldots, C_{n-1}$ is uniquely determined when the arcs C_0 and C_n and the magnitude of the angle formed by the arcs C_0 and C_1 are given, then every absolute conformal invariant of the related horn angle is an absolute conformal invariant of the given rational angle and conversely.* Kasner has shown that a horn angle has only one absolute conformal invariant of finite order besides the magnitude of the angle formed.† It follows from the preceding that a rational angle consisting of the arcs C: $v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\ldots$ and C': $v=v_0+b_1(u-u_0)+b_2(u-u_0)^2+\ldots$, not conformally equivalent to a rectilinear angle, has one and only one absolute conformal invariant of finite order besides the magnitude of the angle formed if $\left(\frac{a_1}{b_1}\right)^{2^p}=1$, $p=0, 1, 2, \ldots$. This result is established for all rational angles in the next section by a direct method.

The result stated on p. 409 may be restated as follows, if we call the arc C_{m+1} in the above the m-th symmetric image of the arc C_0 with respect to the arc C_1:

THEOREM: *A necessary and sufficient condition that a rational angle of the n-th order be conformally equivalent to a pair of intersecting rectilinear segments is that the $(n-1)$-st symmetric image of one of the analytic arcs with respect to the other be the first analytic arc.*

---

* This is the "method of successive symmetries" employed by Kasner, *loc. cit.*, p. 84.

† Kasner, *loc. cit.*, p. 84.

§ 7. *Conformal Invariants of Rational Angles.*

That a rational angle has a unique absolute conformal invariant of higher finite order than the first may be shown directly by using the normal form of such an angle given on p. 417; *i. e.*, the rational angle may be taken to consist of the segment of the line $v=u$ in the vicinity of the origin and the analytic arc

$$
\begin{aligned}
C: \quad v = {} & a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+a_{(2r+1)n+1}u^{(2r+1)n+1}+\dots\\
& +a_{(2r+m)n+1}u^{(2r+m)n+1}+\dots+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\dots,
\end{aligned}
$$

where $a_1^n=1$, $n=$ smallest such positive integer; $a_{rn+1}\neq 0$; and $m=2, 3, \dots, p-1$, $p=$ any positive integer.

Let the analytic arc C be transformed into the analytic arc

$$
\begin{aligned}
C': \quad V = {} & A_1U+A_{n+1}U^{n+1}+A_{2n+1}U^{2n+1}+\dots\\
& +A_{mn+1}U^{mn+1}+\dots+A_{(2r+p)n+1}U^{(2r+p)n+1}\\
& +A_{(2r+p)n+2}U^{(2r+p)n+2}+\dots, \qquad m=3, 4, \dots, 2r+p-1,
\end{aligned}
$$

by the non-singular pseudo-conformal transformation given by

$$
\begin{aligned}
U &= c_1u+c_2u^2+\dots,\\
V &= c_1v+c_2v^2+\dots, \qquad c_1\neq 0.
\end{aligned}
$$

Then the identity follows:

$$
\begin{aligned}
& c_1(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\dots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\dots\\
& \qquad +a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\dots)\\
& +c_2(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\dots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\dots\\
& \qquad +a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\dots)^2\\
& +\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
& \quad =A_1(c_1u+c_2u^2+\dots)\\
& \quad +A_{n+1}(c_1u+c_2u^2+\dots)^{n+1}\\
& \quad +\dots\dots\dots\dots\dots\\
& \quad +A_{mn+1}(c_1u+c_2u^2+\dots)^{mn+1}\\
& \quad +\dots\dots\dots\dots\dots\\
& \quad +A_{(2r+p)n+1}(c_1u+c_2u^2+\dots)^{(2r+p)n+1}\\
& \quad +A_{(2r+p)n+2}(c_1u+c_2u^2+\dots)^{(2r+p)n+2}\\
& \quad +\dots\dots\dots\dots\dots\dots,
\end{aligned}
$$

whence the equalities

$$
\begin{aligned}
c_1a_1 &= A_1c_1 \quad \text{or} \quad a_1=A_1,\\
c_2a_1^2 &= A_1c_2 \quad \text{or} \quad c_2=0,\\
&\dots\dots\dots\dots\dots,\\
c_na_1^n &= A_1c_n \quad \text{or} \quad c_n=0,\\
c_{n+1}a_1^{n+1} &= A_1c_{n+1}+A_{n+1}c_1^{n+1}, \text{ or } c_{n+1} \text{ is arbitrary and } A_{n+1}=0.
\end{aligned}
$$

Assuming that $c_s=0$, $s<kn+1$ and $\neq jn+1$, $k<r$, $j=1, 2, \ldots, k$, and that $A_{jn+1}=0$, it follows that

$$c_{kn+l}a_1^{kn+l}=A_1c_{kn+l}, \qquad l=2, 3, \ldots, n,$$

and

$$c_{(k+1)n+1}a_1^{(k+1)n+1}=A_1c_{(k+1)n+1}+A_{(k+1)n+1}c^{(k+1)n+1}.$$

Thus, in general, $c_s=0$, $s<rn+1$ and $\neq jn+1$, $j=1, 2, \ldots,$ and $A_{kn+1}=0$, $k=1, 2, \ldots, (r-1)$; c_{kn+1}, $k=1, 2, \ldots, (r-1)$, are arbitrary thus far.

Further, we have, on equating the coefficients of u^{rn+1},

$$c_1a_{rn+1}+c_{rn+1}a_1^{rn+1}=A_1c_{rn+1}+A_{rn+1}c_1^{rn+1}$$

or

$$c_1^{rn}A_{rn+1}=a_{rn+1}, \quad \textit{i. e.}, \quad A_{rn+1}\neq 0.$$

Hence, we may take the pseudo-conformal transform of the analytic arc C to be the analytic arc

$$C'': \quad V=a_1U+A_{rn+1}U^{rn+1}+A_{2rn+1}U^{2rn+1}+\ldots+A_{(2r+m)n+1}U^{(2r+m)n+1}+\ldots$$
$$+A_{(2r+p)n+1}U^{(2r+p)n+1}+A_{(2r+p)n+2}U^{(2r+p)n+2}+\ldots,$$
$$m=1, 2, \ldots, p-1.$$

Now, let the non-singular pseudo-conformal transformation, given by

$$U=c_1u+c_2u^2+\ldots,$$
$$V=c_1v+c_2v^2+\ldots, \qquad c_1\neq 0,$$

transform the analytic arc C into the analytic arc C''; then we have the identity

$$\begin{aligned}
&c_1(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad +a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)\\
&+c_2(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad +a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^2\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad =a_1(c_1u+c_2u^2+\ldots)\\
&\quad +A_{rn+1}(c_1u+c_2u^2+\ldots)^{rn+1}\\
&\quad +A_{2rn+1}(c_1u+c_2u^2+\ldots)^{2rn+1}\\
&\quad +\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad +A_{(2r+m)n+1}(c_1u+c_2u^2+\ldots)^{(2r+m)n+1}\\
&\quad +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad +A_{(2r+p)n+1}(c_1u+c_2u^2+\ldots)^{(2r+p)n+1}\\
&\quad +A_{(2r+p)n+2}(c_1u+c_2u^2+\ldots)^{(2r+p)n+2}\\
&\quad +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

whence it follows that

$$\begin{gathered}
c_1a_1=a_1c_1,\\
c_2a_1^2=a_1c_1,\\
\ldots\ldots,\\
c_{rn}a_1^{rn}=a_1c_{rn},\\
c_1a_{rn+1}+c_{rn+1}a_1^{rn+1}=a_1c_{rn+1}A_{rn+1}c_1^{rn+1},
\end{gathered}$$

from which it follows that $c_s=0$, $s\neq kn+1$ and $<rn+1$, while c_{kn+1}, $k=0, 1, 2, \ldots\ldots, r$, are arbitrary thus far (of course, $c_1\neq 0$). Further, assuming that $c_s=0$, $s<(r+k)n+1$ and $\neq jn+1$, $k<r-1$, $j=0, 1, 2, \ldots\ldots$, we have upon equating coefficients of $u^{(r+k)+l}$, $l=2, 3, \ldots\ldots, n$,

$$c_{(r+k)n+l}a_1^{(r+k)n+l}=a_1c_{(r+k)n+l}\,.$$

Consequently, $c_s=0$, $s<(2r+p)+1$ and $\neq jn+1$, $j=0, 1, 2, \ldots\ldots$

The last identity then reduces to

$$\begin{aligned}
&c_1(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\ldots\\
&\qquad\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)\\
&+c_{n+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\ldots\\
&\qquad\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{n+1}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+c_{mn+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\ldots\\
&\qquad\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{mn+1}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad=a_1(c_1u+c_{n+1}u^{n+1}+\ldots\ldots+c_{mn+1}u^{mn+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)\\
&\quad+A_{rn+1}(c_1u+c_{n+1}u^{n+1}+\ldots\ldots+c_{mn+1}u^{mn+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{rn+1}\\
&\quad+A_{2rn+1}(c_1u+c_{n+1}u^{n+1}+\ldots\ldots+c_{mn+1}u^{mn+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{2rn+1}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad+A_{(2r+m)n+1}(c_1u+c_{n+1}u^{n+1}+\ldots\ldots+c_{mn+1}u^{mn+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(2r+m)n+1}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad+A_{(2r+p)n+1}(c_1u+c_{n+1}u^{n+1}+\ldots\ldots+c_{mn+1}u^{mn+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(2r+p)n+1}\\
&\quad+A_{(2r+p)n+2}(c_1u+c_{n+1}u^{n+1}+\ldots\ldots+c_{mn+1}u^{mn+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(2r+p)n+2}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

From this identity the following equalities result:

$$\begin{gathered}
c_1a_1=a_1c_1\,,\\
c_{n+1}a_1^{n+1}=a_1c_{n+1}\,,\\
\ldots\ldots\ldots\ldots\ldots\ldots,\\
c_{(r-1)n+1}a_1^{(r-1)n+1}=a_1c_{(r-1)n+1}\,,\\
c_1a_{rn+1}+c_{rn+1}a_1^{rn+1}=a_1c_{rn+1}+A_{rn+1}c_1^{rn+1},\\
(n+1)\,c_{n+1}a_1^{n}a_{rn+1}+c_{(r+1)n+1}a_1^{(r+1)n+1}=a_1c_{(r+1)n+1}+(rn+1)\,A_{rn+1}c_1^{rn}c_{n+1}\,,
\end{gathered}$$

or

$$(r-1)\,na_{rn+1}c_{n+1}=0,\quad i.\ e.,\quad c_{n+1}=0.$$

Assuming that $c_{kn+1}=0,\ 0<k<r-1$, we get, upon equating coefficients of $u^{(r+k+1)n+1}$,

$$\begin{aligned}
[(k+1)n+1]c_{(k+1)n+1}a_1^{(k+1)n}a_{rn+1}+c_{(r+k+1)n+1}a_1^{(r+k+1)n+1}&\\
=a_1c_{(r+k+1)n+1}+(rn+1)A_{rn+1}c_1^{rn}c_{(k+1)n+1}&,
\end{aligned}$$

or

$$(r-k-1)na_{rn+1}c_{(k+1)n+1}=0, \quad i.\ e., \quad c_{(k+1)n+1}=0, \qquad k=0,1,2,\ldots\ldots,r-2.$$

The above identity may now be written

$$\begin{aligned}
&c_1(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\ldots\\
&\qquad\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)\\
&+c_{rn+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\ldots\\
&\qquad\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{rn+1}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+c_{(r+h)n+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\ldots\\
&\qquad\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(r+h)n+1}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+c_{(2r+p)n+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\ldots\\
&\qquad\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(2r+p)n+1}\\
&+c_{(2r+p)n+2}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\ldots\\
&\qquad\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(2r+p)n+2}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad=a_1(c_1u+c_{rn+1}u^{rn+1}+\ldots\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)\\
&\quad+A_{rn+1}(c_1u+c_{rn+1}u^{rn+1}+\ldots\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{rn+1}\\
&\quad+A_{2rn+1}(c_1u+c_{rn+1}u^{rn+1}+\ldots\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{2rn+1}\\
&\quad+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad+A_{(2r+m)n+1}(c_1u+c_{rn+1}u^{rn+1}+\ldots\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(2r+m)n+1}\\
&\quad+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad+A_{(2r+p)n+1}(c_1u+c_{rn+1}u^{rn+1}+\ldots\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(2r+p)n+1}\\
&\quad+A_{(2r+p)n+2}(c_1u+c_{rn+1}u^{rn+1}+\ldots\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\ldots\\
&\qquad\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots\ldots)^{(2r+p)n+2}\\
&\quad+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\qquad\qquad\qquad h=1,2,\ldots\ldots,r+p-1;\ m=1,2,\ldots\ldots,p-1.
\end{aligned}$$

The following equalities result from this identity:

$$(\alpha)\quad c_1a_{rn+1}+c_{rn+1}a_1^{rn+1}=a_1c_{rn+1}+A_{rn+1}c_1^{rn+1},$$

$$(\beta)\quad c_1a_{2rn+1}+(rn+1)c_{rn+1}a_1^{rn}a_{rn+1}+c_{2rn+1}a_1^{2rn+1}$$
$$=a_1c_{2rn+1}+(rn+1)A_{rn+1}c_1^{rn}c_{rn+1}+A_{2rn+1}c_1^{2rn+1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$(\gamma)\quad c_1a_{(2r+m)n+1}+[(r+m)n+1]c_{(r+m)n+1}a_1^{(r+m)n}a_{rn+1}+\dots\dots+c_{(2r+m)n+1}a_1^{(2r+m)n+1}$$
$$=a_1c_{(2r+m)n+1}+(rn+1)A_{rn+1}c_1^{rn}c_{(r+m)n+1}+\dots\dots+A_{(2r+m)n+1}c_1^{(2r+m)n+1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$m=1,\,2,\,\dots\dots,\,p,$$

where the terms omitted in the last equality do not involve $c_{(r+m)n+1}$ or $A_{(2r+m)n+1}$.* From (α) and (β) it follows that

$$\frac{a_{2rn+1}}{a_{rn+1}^2}=\frac{A_{2rn+1}}{A_{rn+1}^2}.$$

Further, from (γ) we have

$$c_1^{(2r+p)n+1}A_{(2r+p)n+1}=pna_{rn+1}c_{(r+p)n+1}+P(c_1,\,\dots\dots,\,c_{(r+p-1)n+1},\,A_1,\,\dots\dots,\,A_{(r+p-1)n+1}).$$

Since $c_{(r+p)n+1}$ is arbitrary, and since $pna_{rn+1}\neq 0$, $A_{(2r+p)n+1}$ is arbitrary and, therefore, there exists no absolute pseudo-conformal invariant of order $(2r+p)$; *i. e.*, since p is any positive integer, there is no other higher invariant of finite order besides a_1 and $\dfrac{a_{2rn+1}}{a_{rn+1}^2}$.

We have, in the complex plane, the

THEOREM: *Any rational angle not conformally equivalent to a pair of intersecting rectilinear segments has just one absolute conformal invariant of finite order besides the magnitude of the angle.*

§ 8. *Products of Symmetries.*

Consider the symmetry determined by the analytic arc C_1: $v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\dots\dots$, and thus given by

$$S_1:\quad\begin{cases}v=v_0+a_1(U-u_0)+a_2(U-u_0)^2+\dots\dots,\\ V=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\dots\dots,\end{cases}$$

and also the reflection with respect to the line $v=b_0+b_1u$,

$$S:\quad\begin{cases}U=\dfrac{b_0}{b_1}+\dfrac{v}{b_1},\\ V=\ b_0\ +\ b_1u;\end{cases}$$

then $S_1=TST^{-1}$, where T is a non-singular pseudo-conformal transformation which transforms the arc C_1 into a segment of the line $v=b_0+b_1u$.

---

* By equating the coefficients of $u^{(r+h)n+1}$, $h=1,\,2,\,\dots\dots,\,n-1$, only the trivial equalities

$$c_{(r+h)n+1}a_1{}^{(r+h)n+1}=a_1c_{(r+h)n+1}$$

are obtained.

THEOREM: *If S_1 is a symmetry, then $T^{-1}S_1T$ is a symmetry, where T is a non-singular conformal transformation which leaves a point of the determining arc of S_1 invariant.*

For $S_1=T_1ST_1^{-1}$, where T_1 is a non-singular conformal transformation which transforms the determining arc of S_1 into a rectilinear segment through the invariant point and S is the reflection with respect to this segment. Then $T^{-1}S_1T=T^{-1}T_1ST_1^{-1}T=(T_1^{-1}T)^{-1}S(T_1^{-1}T)$, which is a symmetry.

We have also the following simple

THEOREM: *If a non-singular conformal transformation T' which leaves a point invariant is the product of two symmetries S_1, S_2 which leave the same point invariant, then the transform of T' by a non-singular conformal transformation T is the product of the reflection S with respect to a line through the invariant point and a symmetry S_3, where T and S_3 are determined as below.*

For

$$T^{-1}T'T=T^{-1}S_1S_2T=T^{-1}S_1TT^{-1}S_2T=SS_3,$$

where T is a non-singular conformal transformation which transforms the analytic arc which determines the symmetry S_1 into a segment in the vicinity of the invariant point of the line referred to above and S is the reflection with respect to this line.

If the invariant point is the origin, SS_3 is of the form

$$\begin{aligned} U&=a_1u+a_2u^2+\ldots\ldots, \\ V&=b_1v+b_2v^2+\ldots\ldots, \qquad a_1\neq 0, \end{aligned}$$

where the b's are the coefficients of the power series which is the inverse of the power series whose coefficients are the a's, when S is taken as the reflection with respect to the line $v=u$. Again, let S_1 and S_2 be given by

$$S_1:\quad \begin{cases} U=b_1v+b_2v^2+\ldots\ldots=f^{-1}(v), \\ V=a_1u+a_2u^2+\ldots\ldots=f(u), \qquad a_1\neq 0, \end{cases}$$

$$S_2:\quad \begin{cases} U=d_1v+d_2v^2+\ldots\ldots=g^{-1}(v), \\ V=c_1u+c_2u^2+\ldots\ldots=g(u), \qquad c_1\neq 0; \end{cases}$$

then $T'=S_1S_2$ is given by

$$\begin{aligned} U&=g^{-1}[f(u)], \\ V&=g[f^{-1}(v)]. \end{aligned}$$

It is seen that the product of two symmetries which leave the origin invariant is by no means the general non-singular conformal transformation which leaves the origin fixed. In particular, if both symmetries are real, then the real conformal transformation, expressed as a power series of a complex variable, which is their product has unity as the modulus of the coefficient of the first-degree term.

Consider the product of three symmetries S_1, S_2, S_3, determined by three analytic arcs passing through the origin: $S' = S_1 S_2 S_3$. S' is a reversed non-singular conformal transformation, but not, in general, a symmetry. Let $S = T^{-1} S_1 T$, where S is the reflection with respect to the line $u = v$ and T a non-singular conformal transformation which transforms the determining arc of S_1 into the segment in the vicinity of the origin of the line $v = u$, leaving the origin invariant. Then

$$T^{-1} S' T = T^{-1} S_1 S_2 T = T^{-1} S_1 T T^{-1} S_2 T T^{-1} S_3 T,$$

or $S'' = S S_2' S_3'$, where S'', S_2', S_3' are symmetries leaving the origin invariant. Let S_2' be given by

$$\begin{aligned} U &= f^{-1}(v), \\ V &= f(u) = a_1 u + a_2 u^2 + \ldots., \qquad a_1 \neq 0. \end{aligned}$$

and S_3' by

$$\begin{aligned} U &= g^{-1}(v), \\ V &= g(u) = b_1 u + b_2 u^2 + \ldots., \qquad b_1 \neq 0. \end{aligned}$$

Then $S S_2' S_3'$ is given by

$$\begin{aligned} U &= g^{-1}[f(v)], \\ V &= g[f^{-1}(u)]. \end{aligned}$$

Hence, in order that S'' be a symmetry, it is necessary and sufficient that

$$g^{-1} f = f g^{-1},$$

or

$$fg = gf,$$

whence the

THEOREM: *In order that the product of three symmetries determined by three analytic arcs intersecting in the origin be a symmetry, it is necessary and sufficient that the transforms of the second and third factors by a non-singular conformal transformation which transforms the determining analytic arc of the first symmetry into the segment $v = u$, and leaves the origin invariant, have determining analytic arcs which are such that the substitutions represented by the equations of these arcs are commutative.*

Since any symmetry may be transformed by a translation into a symmetry determined by an arc through the origin, the above theorem may be readily modified to apply in the case that the invariant point is not the origin.

The last theorem implies the existence of an infinite set of necessary conditions involving the coefficients of the power series which represent the analytic arcs which determine the symmetries. The most simple necessary condition that the product of three real symmetries which leave a point invariant be a symmetry in the real plane has an intersecting geometrical interpretation. Taking the origin as the invariant point, we have as the three symmetries

$$\begin{aligned} S_1: &\quad Z=\alpha_1\bar{z}+\alpha_2\bar{z}^2+\ldots., \\ S_2: &\quad Z=\beta_1\bar{z}+\beta_2\bar{z}^2+\ldots., \\ S_3: &\quad Z=\gamma_1\bar{z}+\gamma_2\bar{z}^2+\ldots.. \end{aligned}$$

Then $S_1S_2S_3$ is given by

$$Z=\alpha_1\overline{\beta}_1\gamma_1\bar{z}+(\alpha_1^2\overline{\beta}_1^2\gamma_2+\alpha_1^2\overline{\beta}_2\gamma_1+\alpha_2\overline{\beta}_1\gamma_1)\bar{z}^2+\ldots..$$

Let

$$\begin{aligned} C_1: &\quad y=a_1x+a_2x^2+\ldots., \\ C_2: &\quad y=b_1x+b_2x^2+\ldots., \\ C_3: &\quad y=c_1x+c_2x^2+\ldots. \end{aligned}$$

be the analytic arcs which determine S_1, S_2 and S_3 respectively. Then

$$\alpha_1=\frac{1+a_1i}{1-a_1i},\qquad \beta_1=\frac{1+b_1i}{1-b_1i},\qquad \gamma_1=\frac{1+c_1i}{1-c_1i},$$

$$\alpha_2=\frac{2a_2i}{(1-a_1i)^3},\quad \beta_2=\frac{2b_2i}{(1-b_1i)^3},\quad \gamma_2=\frac{2c_2i}{(1-c_1i)^3}.$$

Then

$$\frac{2d_2i}{(1-a_1i)^3(1-b_1i)^3(1-c_1i)^3}=\alpha_1^2\overline{\beta}_1^2\gamma_2+\alpha_1^2\overline{\beta}_2\gamma_1+\alpha_2\overline{\beta}_1\gamma_1,$$

where d_2 is the coefficient of the second-degree term of the power series which represents the analytic arc which determines the resulting symmetry and is, of course, real. On separating into real and imaginary parts, it is found that

$$\frac{a_2(b_1-c_1)}{1+a_1^2}+\frac{b_2(c_1-a_1)}{1+b_1^2}+\frac{c_2(a_1-b_1)}{1+c_1^2}=0,$$

or

$$\mathbf{\Gamma}_1\sin(C_2, C_3)+\mathbf{\Gamma}_2\sin(C_3, C_1)+\mathbf{\Gamma}_3\sin(C_1, C_2)=0,$$

where $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$, $\mathbf{\Gamma}_3$ denote the curvatures at the origin of the arcs C_1, C_2, C_3 respectively, and sin (C_2, C_3) denotes the sine of the angle between the arcs C_2 and C_3, etc. We state the

THEOREM: *If the product of three real symmetries determined by arcs through a common point is also a symmetry, then the sum of the three products of the curvature at the common point of the arc determining one factor by the sine of the angle between the two arcs determining the other symmetries, and oriented according to the order of the factors in the product, is equal to zero.*